Prehistory of Faà di Bruno's Formula

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1. INTRODUCTION: EXPANSION OF COMPOSITE FUNCTIONS. Recent papers [**15**], [**17**], [**22**] have explored the work of Faà di Bruno and his near-contemporaries on the series expansion of composite functions. Here, earlier work unnoticed in these papers is described. These anticipate many of the results attributed to others. This earlier work originates with Arbogast, with later reworkings by Knight, West, De Morgan, and others.

The series expansion in powers of x of a suitably-differentiable composite function $g(f(x))$ has the form

$$\sum x^m \mathrm{D}^m \big[g\big(f(x)\big)\big]_0 / m!, \tag{1}$$

where D^m denotes the mth x-derivative of the composite function and the subscript 0 means evaluation at $x = 0$. Each of these derivatives is expressed by Faà di Bruno's formula as a sum of terms that comprise derivatives of g multiplied by terms involving the derivatives of f. Francesco Faà di Bruno gave the formula in two short papers [**13**], [**14**] of 1855 and 1857, without proof or stated sources. It is

$$\mathrm{D}^m\big[g\big(f(x)\big)\big] = \sum \frac{m!}{b_1!b_2!\dots b_m!} g^{(k)}\big(f(x)\big) \left(\frac{f'(x)}{1!}\right)^{b_1} \left(\frac{f''(x)}{2!}\right)^{b_2} \cdots \left(\frac{f^{(m)}(x)}{m!}\right)^{b_m}, \tag{2}$$

where the sum is over all possible combinations of nonnegative integers $b_1, b_2, \dots, b_m$ such that $b_1 + 2b_2 + \cdots + mb_m = m$ and $k = b_1 + b_2 + \cdots + b_m$. The $g^{(k)}(y)$ denote kth y-derivatives of $g(y)$, and the $f^{(j)}(x)$ are jth x-derivatives of $f(x)$. Faà di Bruno also gave an alternative version of (2) as an $n \times n$ determinant. A full account of the formula, and its variants, is given in the papers [**15**], [**17**], [**22**].

Johnson [**22**] discovered that Faà's formula was anticipated by thirty-six years by Silvestre Lacroix, in volume 3 (1819) of the second edition of his *Traité du Calcul Différential et du Calcul Intégral* [**31**]; and by thirty-two years in an 1823 dissertation by Heinrich Ferdinand Scherk [**36**]. Further, both Johnson and Gould [**17**] describe related work by Hoppe [**21**], "T.A." [**1**], and "A." [**2**] that also precede that of Faà di Bruno by some years. "T.A." and "A." are one and the same person, identified by Johnson as an artillery captain, J.F.C. Tiburce Abadie, who around 1850–55 published some papers in the *Nouvelles Annales de Mathématiques*. Gould notes that the *Index to the Royal Society Catalogue of Scientific Papers 1800–1900* (page 256 of the *Mathematics* volume) identifies "T.A." as Théodore Anne, but this is certainly wrong. Théodore Anne (1797–1869) was a historical writer, novelist, and dramatist who appears to have published nothing on science or mathematics. Johnson also mentions, but does not do justice to, the much earlier work of L. F. A. Arbogast in his *Traité du Calcul des Dérivations* [**3**]. Here, we summarise Arbogast's work more fully, and we examine some later reworkings of it by British authors that predate many of the works just mentioned.

2. ARBOGAST'S *CALCUL DES DÉRIVATIONS* (1800). Louis François Antoine Arbogast (1759–1803) deserves credit for discovering rules for forming the various derivatives $D^m[g(f(x))]_0$. Not only did he state the formula usually attributed to Faà di Bruno, but he described algorithms, as lengthy and wordy *Règles* (rules), that are equivalent to it and are easier to use in applications. He established many relations between his coefficients and derivatives, which readily enabled higher-order terms to be calculated from preceding ones. In so doing, he created a sort of algebra of differential operators that was a precursor of automatic differentiation; and he introduced the notion of differential operators with *negative* integer indices $-m$ corresponding to m repeated integrations. Arbogast claimed, with some justice, that his work was a generalisation of the differential calculus, rather than merely an application of it.

Arbogast went on to consider composite functions of the forms $h[f(x), g(x)]$ and $k[f(x), g(x), h(x)]$ as well as composite functions of two or more independent variables, such as $g[f(x, y)]$, $g[f(x, y, z)]$, $h[f(x, y), g(x, y)]$. These topics were not considered by Faà di Bruno, but they were re-examined in some of the earlier British works discussed here.

Throughout, Arbogast supposes that the "inner" functions are already expanded as polynomials in the independent variables. Thus, instead of $g[f(x)]$, he uses $\phi(\alpha + \beta x + \gamma x^2 + \delta x^3 + \cdots)$, and his *Règles* consist of instructions for raising or lowering powers and numerical coefficients, and for changing letters of the Greek alphabet into their neighbours. Though hardly elegant mathematics, the rules are effective in generating derivatives to any desired order.

It must also be said that his treatise is not particularly well written: the organisation of its parts is loose and sometimes illogical, his notation is often ill-chosen, and he gives few practical examples that would have emphasised the utility of his methods. Perhaps the most directly useful part of the whole treatise is a table [**3**, pp. 28–29] that allows easy construction of the sums for the first ten derivatives $D^m[g(f(x))]_0$. This table shows the coefficients, as functions of $\alpha, \beta, \gamma, \ldots,$ of the various derivatives of ϕ. These are normally more than would be required for practical numerical calculation to high accuracy. In Figure 1, we show here for the sake of illustration just the last term, which gives the coefficient of x^{10} in the expansion.

The first four terms are

$$\phi(\alpha + \beta x + \gamma x^2 + \delta x^3 + \cdots) = \phi(\alpha) + x.D\phi.\beta + x^2\{D\phi.\gamma + D^2\phi.(\beta^2/2)\}$$
$$+ x^3\{D\phi.\delta + D^2\phi.(\beta\gamma) + D^3\phi.(\beta^3/6)\} + \cdots, \quad (3)$$

where $D^r\phi$ denotes the rth derivative of $\phi(y)$ evaluated at $y = \alpha$. Following Arbogast's rules for changing powers and letters, it is straightforward to extend his table to any desired accuracy. By analogy with Taylor's series, it is convenient to define the rth *polynomial derivative* as the coefficient of x^r multiplied by $r!$ (this expression, corresponding to Arbogast's *dérivation*, was introduced by J. West; see section 4). Arbogast identified these polynomial derivatives by the notation $D^r.\phi\alpha$, where $\phi\alpha$ means $\phi(\alpha)$ and $D^r.\phi\alpha$ means $D^r.\phi$ evaluated at α. Here, only the dot distinguishes polynomial differentiation from the ordinary rth derivative of ϕ, which appears as $D^r\phi\alpha$, so careful reading is necessary! In addition, he often places a small letter "c" below a D^r to indicate division by the factorial $r!$

One of his key results [**3**, p. 34] is the expression of each coefficient of x^n in the final expansion (1) or (3) as the sum

$$\underset{c}{D}^n.\phi\alpha = \underset{c}{D}^n\phi\alpha.\beta^n + \underset{c}{D}^{n-1}\phi\alpha.D.\beta^{n-1} + \underset{c}{D}^{n-2}\phi\alpha.\underset{c}{D}^2.\beta^{n-2} + \cdots + D\phi\alpha.\underset{c}{D}^{n-1}.\beta. \quad (4)$$

$$
\begin{aligned}
&+ \mathrm{D}\varphi\alpha \cdot \lambda \\
&+ \frac{\mathrm{D}^2\varphi\alpha}{1.\,2} \cdot \{2\beta\kappa + 2\gamma\iota + 2\delta\theta + 2\varepsilon\eta + \zeta^2\} \\
&+ \frac{\mathrm{D}^3\varphi\alpha}{1.\,2.\,3} \cdot \left\{\begin{matrix} 3\beta^2\iota + 3\beta(2\gamma\theta + 2\delta\eta + 2\varepsilon\zeta) \\ + 3\gamma^2\eta + 3\gamma(2\delta\zeta + \varepsilon^2) + 3\delta^2\varepsilon \end{matrix}\right\} \\
&+ \frac{\mathrm{D}^4\varphi\alpha}{1 \ldots 4} \cdot \left\{\begin{matrix} 4\beta^3\theta + 6\beta^2(2\gamma\eta + 2\delta\zeta + \varepsilon^2) \\ + 4\beta(3\gamma^2\zeta + 3\gamma 2\delta\varepsilon + \delta^3) + 4\gamma^3\varepsilon + 6\gamma^2\delta^2 \end{matrix}\right\} \\
&+ \frac{\mathrm{D}^5\varphi\alpha}{1 \ldots 5} \cdot \left\{\begin{matrix} 5\beta^4\eta + 10\beta^3(2\gamma\zeta + 2\delta\varepsilon) \\ + 10\beta^2(3\gamma^2\varepsilon + 3\gamma\delta^2) + 5\beta 4\gamma^3\delta + \gamma^5 \end{matrix}\right\} \\
&+ \frac{\mathrm{D}^6\varphi\alpha}{1 \ldots 6} \cdot \left\{\begin{matrix} 6\beta^5\zeta + 15\beta^4(2\gamma\varepsilon + \delta^2) + 20\beta^3 3\gamma^2\delta \\ + 15\beta^2\gamma^4 \end{matrix}\right\} \\
&+ \frac{\mathrm{D}^7\varphi\alpha}{1 \ldots 7} \cdot \{7\beta^6\varepsilon + 21\beta^5 2\gamma\delta + 35\beta^4\gamma^3\} \\
&+ \frac{\mathrm{D}^8\varphi\alpha}{1 \ldots 8} \cdot \{8\beta^7\delta + 28\beta^6\gamma^2\} \\
&+ \frac{\mathrm{D}^9\varphi\alpha}{1 \ldots 9} \cdot 9\beta^8\gamma \\
&+ \frac{\mathrm{D}^{10}\varphi\alpha}{1 \ldots 10} \cdot \beta^{10}
\end{aligned}
\quad \Bigg|\; x^{10}
$$

Figure 1. A portion of Arbogast's table [**3**, pp. 28–29], showing the coefficient of x^{10} of the composite expansion.

This gives each polynomial derivative of ϕ as a sum of terms involving each ordinary ϕ-derivative multiplied by polynomial derivatives of powers of β, where "β" is shorthand for the function $\beta + \gamma x + \delta x^2 + \epsilon x^3 + \cdots$. The various polynomial derivatives of powers of the function "β" are then easily built up, either from Arbogast's table or by application of simple rules.

The end result is of course identical with that expressed by Faà di Bruno's formula. Furthermore, result (4) is *precisely* the 1850 formula of "T.A." (see Johnson [**22**, p. 223]), when notational changes are taken into account. An equivalent result, which also predates that of "T.A.", was given by J. West.

Almost as an afterthought, Arbogast states, clearly and explicitly, “Faà di Bruno’s formula,” fifty-five years before Faà di Bruno did so. A partial translation of Arbogast’s “Remarques” [**3**, pp. 43–44] is:

> The polynomial quantities β^n, $\mathrm{D}.\beta^{n-1}$, $\mathrm{D}_c^2.\beta^{n-2}$, etc. can also be formed by combinations, though in a manner less easy and less analytical than by derivations; so it will not be useless to make known the agreement of polynomial quantities with certain results which one obtains by combinations, so that, on occasion, one can compare our formulae with the theorems which various analysts have arrived at by combinations or by other means. . . .
>
> Thus, taking the sign **f** to designate an assemblage of terms or products added one to another, one can represent the coefficient of x^m in the n'th power of a polynomial by
>
> $$\mathbf{f}.\frac{n(n-1)(n-2)\ldots.(p+2)(p+1)}{1.2\ldots.q\times 1.2\ldots.r\times 1.2\ldots.s\times\ldots}.\beta^p.\gamma^q.\delta^r.\varepsilon^s\ldots.$$
>
> $$\ldots.\text{ where } p+q+r+s=n,$$
>
> $$\text{and } q+2r+3s=m;$$
>
> . . . which one can restate in the following manner.
>
> The letters β, γ, δ, etc. having as indices or respective quantifiers 1, 2, 3, etc., that is to say, these letters being represented by α_1, α_2, α_3, etc.; . . . $\mathrm{D}_c^m.\beta^n$ denotes the sum of all the different products each composed of n letters (the same letters being repeated), under the condition that in each product the sum of the indices of all the letters be equal to $n+m$, and that one gives as numerical coefficient to each of these products the quantity that expresses the number of permutations that the letters which form the product can allow, on having regard for repeated letters.
>
> By means of these indices, the investigation of products can be re-expressed as this question on the partition of numbers: knowing the whole number $n+m$, find all the different ways of forming it by addition of n smaller whole numbers.

This, allied with (4), gives the required general result. On permutations, Arbogast cites both Jacques Bernoulli’s *Ars Conjectandi* and Hindenburg’s *Infinitinomii dignitatum historia*. . . , but not the work of De Moivre.

Though quite well known in his own day, Arbogast was not part of the influential Paris scene. He worked mainly in Strasbourg, in Alsace, as a professor of mathematics. In 1791 he was appointed rector of the University of Strasbourg, and he was elected to the Legislative Assembly. Just a year later, he was elected to the Académie des Sciences. His closest mathematical associates were the brothers F. J. and J. F. Français (see Grattan-Guinness [**18**, pp. 211–217]). The *Calcul des dérivations* was known to Lagrange, who mentions it in his 1797 *Théorie des fonctions analytiques* [**32**] as being still in manuscript.

In the second edition of his large three-volume treatise on calculus, Sylvestre Lacroix [**31**, vol. 1 (1810), p. *xxx*, pp. 315–326] summarises Arbogast’s work (but there is no mention of it in his first edition of 1797–1800). He explicitly states his wish to mark the place that the researches of Arbogast and of Christian Kramp *ought* to hold in transcendental analysis, though then little studied in France. But he also comments unfavourably on the numerous notations that Arbogast employed, and bases his account on an 1806 paper by the Italian Pietro Paoli [**35**]. Finally, among copious *Corrections et Additions* in his third volume, Lacroix states the Faà di Bruno formula [**31**, vol. 3 (1819), p. 629]. Kramp’s 1808 *Élemens d’Arithmétique universelle* [**30**, pp. 271–300] also gives an account of Arbogast’s theory, including a rather clumsy statement of Faà di Bruno’s formula on pages 278–9.

More surprisingly, Arbogast’s *Calcul des dérivations* was known to Robert Woodhouse of Cambridge in time for him to give an account of it in his 1803 *The Principles* *of Analytic Calculation* [**41**]. There, in pages 109–132, he gives the first description in English of Arbogast's method, deducing equation (4) and applying the method to several examples. These include the expansion of powers, exponentials, and logarithms of a polynomial, and products and quotients of polynomials. For expanding the power of a polynomial, he mentions that, as early as 1697, Abraham De Moivre [**7**] correctly gave "the law of the coefficients ... in many words and with little simplicity"; but now Arbogast's improved notation readily allowed such solutions to be "rigorously effected." In fact, De Moivre's account is fairly readable, and he states clear rules that can be used to construct the coefficients $\underset{c}{\mathrm{D}}^r.\beta^{n-r}$ in Arbogast's result (4).

Several writers have remarked on both the merits and the deficiencies of Arbogast's presentation. In a lengthy 1838 survey for the *Encyclopaedia Britannica*, the Edinburgh-based Scot John Leslie wrote equivocally [**33**, pp. 601–602]:

> But though the method of Derivations should not possess that logical superiority over the Fluctionary or Differential Calculus which its author so fondly supposes, yet is the invention entitled to the highest praise for its beautiful perspicuity and its ready and most extensive application. We have only to regret that it has required a new system of characters, when the ordinary notation has become so familiar, and attained so great perfection. Such mutations, like the diversity of languages, may be deemed a serious evil, since they divert the attention to the mere accesaries [*sic*] of learning, and retard or obstruct the acquisition of real knowledge.

But Arbogast's treatise impressed John Herschel, who wrote the following in the rival *Edinburgh Encyclopaedia* [**20**, p. 376]:

> Arbogast has considered the subject ... in his *Calcul des Derivations*, (1800) and his methods, which include in simple and expressive formulae the co-efficients of the most gigantic developements [i.e. series expansions], seem to unite the two great requisites, elegance and power, in a more perfect manner than could have been expected.

Herschel's friend Charles Babbage also knew Arbogast's work, and he may have intended to use the methods for numerical calculations by his ill-fated "Analytical Engine." To do so, he would have set $x = 0.1$, making $\alpha, \beta, \gamma, \dots$ the successive digits of decimal numbers. Babbage discussed his engine with Ada Lovelace, who translated L. F. Menabrea's account of it and added copious notes. In one of these notes, she wrote: "The methods of Arbogast's *Calcul des Dérivations* are peculiarly fitted for the notation and the processes of the engine. Likewise the whole of the Combinatorial Analysis ... " [**34**, p. 724].

In more recent times, F. Cajori observed that, with Arbogast's treatise: "It looked indeed as if the different mathematical architects engaged in erecting a proud mathematical structure found themselves confronted with the curse of having their sign language confounded so that they could less easily understand each other's speech" [**4**, pp. 225–226]. (See also Grattan-Guinness [**18**, vol. 1, pp. 211–217], who notes both the power and laboriousness of Arbogast's procedures.)

In 1891 two articles by H. W. Lloyd Tanner [**38**], [**39**] gave useful brief summaries of the work of Arbogast and of several later accounts of his methods. Among the latter are early papers by Paoli [**35**] and Arbogast's associate J. F. Français [**16**], which aimed to improve on Arbogast's notation. Paoli's reworking, which was used by Lacroix, added nothing new and does not state Faà di Bruno's formula. But Français introduced the idea of differentiating with respect to the various constants $\beta, \gamma, \delta, \dots$ in his rules for deriving one polynomial derivative from the next. (Français's idea was exploited more fully by De Morgan; see section 5.) Of the works discussed here, Tanner did not know that of Knight, and he gives West less than his due; but he usefully summarises work of De Morgan and Donkin, and also that of some later authors, including Arthur Cayley and Samuel Roberts.

One of very few modern authors aware of Arbogast's priority over Faà di Bruno is Donald Knuth [**29**, pp. 52, 481–483]. He correctly attributes the result to Arbogast, mentioning that it "was forgotten for many years, then rediscovered independently by F. Faà di Bruno . . . who observed that it can also be expressed as a determinant" [**29**, p. 483].

3. THE WORK OF THOMAS KNIGHT. In a long paper [**23**] communicated by Humphry Davy to the Royal Society of London in 1810, the Englishman Thomas Knight addressed the "expansion of multinomial functions." This topic had, in Knight's words [**23**, p. 49]:

> of late, been so ably and fully treated by M. Arbogast . . . Nevertheless, as he is the only one that has hitherto cultivated this part of analysis with any *great* success . . . I hope it will not be thought superfluous if I show how the same things may be accomplished in a very different manner. . . . By the procedure here made use of, we shall be enabled to arrive at many new and remarkable theorems . . . which could not, I imagine, be very easily found by M. Arbogast's methods.

Between 1811 and 1817, Knight published seven mathematical papers in the *Philosophical Transactions of the Royal Society*. Around the same time, he published three papers in *Nicholson's Journal*, and several in *Leybourn's Mathematical Repository* (the latter not noted in the *Royal Society Catalogue of Scientific Papers, 1800–1900*). The *Repository* was based at the Royal Military College, Marlow (later Sandhurst) and was at that time mainly compiled by James Ivory and William Wallace. These two tried hard to popularise French analytical methods, and so would have welcomed Knight's several short articles on series expansions (see Craik [**6**]). Some of these were rather trivial, such as his rederivations of exponential and trigonometric series [**24**], [**27**]. But others were more difficult, such as the trigonometric expansion of functions $f(y)$, where $y = (A + c\cos x)^m$ [**26**], and two short letters about his re-examination of Arbogast's work [**25**].

In his Royal Society paper [**23**], Knight constructs an elaborate notation, with subscripts, superscripts, upper dashes, lower dashes, and overdots, that is even more repellant than Arbogast's. He even uses a curious mixture of old-fashioned Newtonian fluxions and fluents, together with Leibnizian integral signs, which adds to the modern reader's confusion. But there is no doubt that Knight understood what he was doing and recovered many of Arbogast's results in his own way. As Arbogast had done, he gives rules in words for obtaining his various coefficients, and he lists the first few of these, noting agreement with Arbogast. Also like Arbogast, he considers more complicated cases, such as the expansion of functions of the form

$$\phi\left\{F(c + c'x + c''x^2 + \cdots),\, f(e + e'x + e''x^2 + \cdots),\ \text{\&c.}\right\}$$

and functions of polynomials in two or more independent variables x, y, etc. As he stresses the central role of "inverse derivation," he seems, like Arbogast, to have felt comfortable with differential operators and their inverses. Though it cannot be said that this paper is easy reading, it is commendable as a rather early attempt by a little-known British mathematician to engage with up-to-the-minute French analysis.

Knight begins by observing: "If $f(c + z)$ represent any function of $c + z$, and the fluxions be taken, separately, with respect to c and z, the fluxional coefficient is the same in both cases" [**23**, p. 50]. On taking z to be $c'x + c''x^2 + c'''x^3 + \cdots$, he quickly shows that the desired expansion

$$f(c + c'x + c''x^2 + c'''x^3 + \cdots) = \mathrm{B} + \underset{1}{\mathrm{B}}\,x + \underset{2}{\mathrm{B}}\,x^2 + \underset{3}{\mathrm{B}}\,x^3 + \cdots + \underset{n}{\mathrm{B}}\,x^n + \ \&\text{c.}$$

must equal the fluent (i.e., the integral) with respect to x of the product of two series:

$$(c' + 2c''x + 3c'''x^2 + \cdots + nc''^{\cdots n}x^{n-1} + \cdots)\left(\mathrm{B}' + \underset{1}{\mathrm{B}}'x + \underset{2}{\mathrm{B}}'x^2 + \underset{3}{\mathrm{B}}'x^3 + \cdots\right),$$

where dashes on the Bs "represent the fluxional coefficients [i.e., derivatives] ... with respect to c." (In fact, Knight places all his dashes *above* the symbols c and B, rather than superscripts as here. Note, too, that dashes on cs do not represent differentiation since these are constants.) Comparison of coefficients of powers of x gives the sequence of results

$$\underset{1}{\mathrm{B}} = c'\mathrm{B}', \qquad \underset{2}{\mathrm{B}} = \frac{2c''\mathrm{B}' + c'\underset{1}{\mathrm{B}}'}{2}, \qquad \underset{3}{\mathrm{B}} = \frac{3c'''\mathrm{B}' + 2c''\underset{1}{\mathrm{B}}' + c'\underset{2}{\mathrm{B}}'}{3}, \qquad \ldots,$$

where $\mathrm{B} = f(c)$ and $\mathrm{B}' = f'(c)$. From these, the B_n can be found sequentially.

In fact, this method of comparing two forms of a derivative resembles one employed long before by Euler to obtain series expansions of particular functions for which Taylor's series is inconvenient. Euler's examples include the mth power and the exponential of any polynomial [**12**, pp. 519, 535]. Other such examples are given by D. F. Gregory [**19**, pp. 70–72].

But Knight rightly observes that his result "affords by no means an easy way of calculating the coefficients," and he goes on to consider three separate methods for deriving a coefficient $\underset{n}{\mathrm{B}}$ from the preceding $\underset{n-1}{\mathrm{B}}$. These yield some of Arbogast's rules, again stated in words. Next, he derives the rule for finding $\underset{n-m}{\mathrm{B}}$ from $\underset{n}{\mathrm{B}}$. Then he poses a "*Problem. It is required to find* $\underset{n}{\mathrm{B}}$ *without knowing any of the coefficients that precede or follow it*" [**23**, p. 60]. His solution, though not very clearly stated, amounts precisely to Faà di Bruno's formula (2) [**23**, pp. 60–61]:

> It is, in the first place, evident enough, from what has been done, that
>
> $$\underset{n}{\mathrm{B}} = f'(c)c''^{\cdots n} + f''(c)\psi'' + f'''(c)\psi''' + \ldots. + f''^{\cdots(m-1)}(c)\psi''^{\cdots(m-1)}$$
> $$+ f''^{\cdots m}(c)\psi''^{\cdots m} + \ldots\ldots. + f''^{\cdots n}(c)\frac{c'^n}{2.3.4\ldots.n}$$
>
> where $\psi''^{\cdots m}$ consists (without considering denominators) of all the combinations that can be found of c', c'', c''', &c. in which the sum of strokes shall be n,* and the sum of the exponents m. [Footnote *: I mean when the powers are expanded, as when c'^3 is written $c'c'c'$.] But to form these combinations, for the higher powers, would not be very easy. It may not be amiss to inquire, therefore, for some regular method of immediately deriving $\psi''^{\cdots(m-1)}$ from $\psi''^{\cdots m}$; so that we may get all the ψ's successively, beginning with $c'^n/2.3\ldots.n$ which multiplies $f''^{\cdots n}(c)$.
>
> I shall take no notice of any numbers, which divide the different terms, till the end of the operation; having shown, in Article 5, that it will be sufficient then to place the product $2.3.4\ldots.\mu$ under every μth power.

(Note that $c''^{\cdots n}$ means c with n dashes, but c'^n means the nth power of c'.)

This seems to be the first statement in English of Faà di Bruno's formula. Knight's reference to "all the combinations ..." shows that he was aware that it may be derived by combinatorial arguments. Though Faà di Bruno did not show how his equivalents of the ψ-coefficients might more easily be obtained than by constructing all the required combinations, Knight provides a rule in words [**23**, p. 62] similar to one of Arbogast's. He then gives a practical example with $n = 10$, finding $\psi''^{...9}$ and $\psi''^{...8}$ by working back from $\psi''^{...10} = c'^{10}/2.3.4 \ldots 10$.

The rest of Knight's paper concerns the expansion of functions of two or more polynomials in x and the expansion of functions of polynomials in two independent variables x and y. For these, Knight believed his methods to be superior to Arbogast's. Needless to say, his notation becomes yet more complicated, and mercifully need not be discussed here.

Thomas Knight is a rather shadowy figure. Despite his interests in mathematics and science, he did not become a Fellow of the Royal Society, and he does not appear in any biographical dictionary. As well as the papers already mentioned, he wrote a rather polemical pamphlet criticising Laplace's theory of capillary action, in which the author is given as "T.K." [**28**]. In his mathematical papers, Thomas Knight gives his address as Papcastle, in Cumberland; and this pamphlet was printed at nearby Cockermouth. Though several men of this name studied at Cambridge or Oxford, all can be ruled out. One person of that name who was admitted to Trinity College, Cambridge, in 1816 and who later pursued a legal career was a son of "our" Thomas.

A search of genealogical records has recently been made by Ms. Gloria Edwards of Cockermouth, to whom the author is most grateful. These show that Thomas Knight was born in 1775 and died in 1853. His parents were John Knight and Henrietta Cunyngham. The family was rich, and the property at Papcastle included a large plot of land. Another family home was at Henley Hall in Shropshire, given as the address of fourteen children of Thomas Knight and his wife Isabella Walker.

Knight's mathematical papers were little noticed in his own day or afterwards. Augustus De Morgan, though an avid bibliophile, did not know of Knight's papers when he wrote his own 1846 account of Arbogast's theory. And John West's version, probably written around the same time as Knight's, is certainly independent of it.

4. THE TREATISES OF JOHN WEST. The *Mathematical Treatises* of John West (1756–1817) [**40**] seem to have been written mostly between 1800 and 1810, but they were not published until 1838, long after West's death in 1817. West had been a student and later assistant at St Andrews University in Scotland. However, despite his obvious talent, he was obliged to emigrate to Jamaica to earn a living. There he became an Anglican priest and pursued mathematics as a recreation (see Craik [**5**]).

At the time they were written, West's *Treatises* were the first fairly complete account in English of calculus in a Lagrangian style; and, unlike Lagrange, he wrote with great clarity. But tardy publication robbed him of influence and credit. Less concerned with expounding the rigorous foundations of analysis, he gave many practical examples of its applications to astronomy and to the compilation of mathematical tables. He also incorporated a rather full account of Arbogast's methods, but without mentioning the latter by name, and he completely reworked the theory in his own better notation [**40**, vol. 1, pp. 88–180]. Unlike Arbogast and Knight, he restricts attention to functions of a single polynomial expansion: $\phi(\alpha + \beta x + \gamma x^2 + \delta x^3 + \cdots)$.

Like Arbogast, West derives a version of equation (4). As we have seen, this is also the formula of "T.A." He gives a table of the first several terms, up to the ninth derivative [**40**, vol. 1, pp. 106–106]: this is equivalent to, but more compact than Arbogast's, but the latter gives one more term (see Figure 1; West's table is reproduced in [**5**, p. 59]). Rules for constructing the coefficients to any degree are clearly set out in words. Though less compact than Faà di Bruno's formula (which West does not state explicitly), the table and rules are very convenient for practical calculations.

Again like Arbogast, West derives many results connecting the various coefficients of the expansion: several of these are not in Arbogast's treatise. In particular, he establishes connections between the polynomial derivatives of integer powers of the series $\beta + \gamma x + \delta x^2 + \cdots$ (which are needed in his version of equation (4)) and the polynomial derivatives of powers of $\alpha + \beta x + \gamma x^2 + \delta x^3 + \cdots$. Having done so, he deduces his Theorem 8 [**40**, vol. 1, p. 138], which is exactly the result known as *Hoppe's formula*, described in [**22**, p. 224] and [**17**, p. 98]. Hoppe's work dates from 1845 [**21**], at least thirty and perhaps as much as forty years after West wrote this part of his *Treatises*.

A streamlined version of "Hoppe's formula" is that designated *Scott's formula* by Johnson [**22**, pp. 226–227]. This appears in an 1861 paper [**37**] by George Scott of Trinity College, Dublin. Johnson wrongly claims this as the first work in English to contain either of the formulas of Faà di Bruno or Hoppe. Scott's result incorporates the fact that, in the notation of (2), no power of the inner function $f(x)$ can appear in the coefficients multiplying the various derivatives $g^{(k)}$. Thus Hoppe's formula contains terms that in fact cancel, and these terms can be eliminated by artificially setting $f(0)$ (Arbogast's and West's α) to zero in the sum for each coefficient. But West seems not to have noticed this simplification.

West's several numerical examples and applications illustrate the power of Arbogast's method in quickly generating good approximations. Anyone who tries to apply the method in this way will soon find that West's or Arbogast's tables of coefficients are far more useful than Faà di Bruno's general formula, whatever other benefits the compactness of the formula may have. A fuller account of West's reworking of Arbogast's theory appears in [**5**], and further recapitulation seems unnecessary. All that needs to be said is that West's hard-to-find and little-known *Treatises* remain the best account in English of Arbogast's *Calcul des dérivations*, clearer than the presentation of Arbogast himself but restricted to functions of a single polynomial in x.

5. AUGUSTUS DE MORGAN'S CONTRIBUTIONS. Long after West's *Treatises* were written, but still nine years before Faà di Bruno's first paper stating his formula, Augustus De Morgan published in 1846 a paper entitled "On Arbogast's Formulae of Expansion" [**9**]. On page 238, De Morgan pays tribute to West's "very complete attempt, as far as the series of one variable is concerned," and mentions that he had already considered the matter, in ignorance of West's work, in his book *The Differential and Integral Calculus* [**8**] of 1842 (or rather 1839, when this part of the book first appeared separately). Tanner [**39**, p. 92] objects to "serious defects" of exposition, but notes "how largely it is due to De Morgan's enthusiastic recommendations of the method that it has not passed out of memory."

In his book, De Morgan employs Français's device of differentiating with respect to constant coefficients, and in his 1846 paper he develops this idea further. There he formalises the manipulative operations that yield the coefficients of powers of x in expansions of functions $\phi(a + bx + cx^2 + ex^3 + \cdots)$, envisaging a set of operations, $\alpha, \beta, \gamma, \epsilon, \ldots$ and their inverses that act only on the respective letters $a, b, c, e, \ldots$. (De Morgan omits d and δ from his alphabets.) Though De Morgan's account is rather unclear, Arbogast's "polynomial derivatives" are obtained by successively applying

$$\mathrm{D.} = \alpha\beta^{-1}, \qquad \mathrm{D}^2. = (\alpha\beta^{-1} + \beta\gamma^{-1})\alpha\beta^{-1} = \alpha^2\beta^{-2} + \alpha\gamma^{-1},$$

$$\mathrm{D}^3. = (\alpha\beta^{-1} + \beta\gamma^{-1} + \gamma\epsilon^{-1})(\alpha^2\beta^{-2} + \alpha\gamma^{-1}) = \alpha^3\beta^{-3} + \alpha^2\beta^{-1}\gamma^{-1} + \alpha\epsilon^{-1}, \quad \text{etc.}$$

to $\phi(a)$, where $\beta^{-1}, \gamma^{-1}, \ldots$ denote integrations with respect to $b, c, \ldots,$ and α denotes differentiation with respect to a.

He also gives a combinatorial analogue of his processes, which involves moving counters in a series of boxes according to set rules. (In the interest of brevity, the reader is referred to De Morgan's paper for details.) Each possible nth state resulting from n successive moves is equivalent to one term of the expression for the nth x-derivative of the composite function. He gives combinatorial rules for constructing each $(n+1)$th state from each nth state, and he arrives at Arbogast's result, here given in equation (4).

De Morgan observes [**9**, p. 245] that the "values of $\mathrm{D}^m b^n$ [equivalent to Arbogast's $m!\,\underset{c}{\mathrm{D}}^m.\beta^{n-m}$] are formed with very little practice as fast as they can be written." He shows, by his combinatorial argument, that "the coefficient which multiplies $b^p c^q e^r$ &c. in any derivative of b^n, is evidently

$$\frac{1.2.3\ldots n}{1.2.3\ldots p \times 1.2.3\ldots q \times 1.2.3\ldots r \times \&\text{c.}}."$$

This is in accordance with Faà di Bruno's formula and also the much earlier results of De Moivre and Thomas Knight, as already mentioned. De Morgan then extends his method to functions of polynomials in two independent variables. De Morgan returned to his combinatorial argument in a further paper of 1851 [**10**], and in the same volume a paper by Donkin [**11**] gave another version of De Morgan's operator methods (see also Tanner [**39**]). Clearly, the modern combinatorial derivations of Faà's formula discussed by Johnson [**22**, pp. 218–219] have much earlier antecedents in Arbogast, Knight and, especially, De Morgan.

De Morgan had no doubts of the practical advantages of Arbogast's method: "With respect to the power of this method, none can judge but those who have tried both it and substitutes for it. There is no producing conviction of the superiority of any process by description." As an illustration, he urges his readers to try the simple expression $(a + bx + cx^2)^5$, expanded by both the binomial theorem and by Arbogast's method, and to "decide the question for himself" [**9**, pp. 246–247].

6. CONCLUSION. Though mathematics in Britain was at a low ebb in the early decades of the nineteenth century, we have seen how three British mathematicians—Knight, West, and De Morgan—separately reworked Arbogast's *Calcul des dérivations*. They obtained results equivalent to Faà di Bruno's formula and to other formulas later rediscovered by others. It is unlikely that Faà di Bruno knew the older works in English. But he could have seen De Morgan's papers. Moreover, it is very possible that he was familiar with Arbogast's book and with Paoli's paper, though he made no mention of them.

Faà di Bruno gave no proof of his formula. Tanner [**39**, p. 86] agrees with De Morgan [**9**, pp. 239, 241] that Arbogast obtained his rules "partly by inspection" and "upon a process of observation." None of Knight, West, or De Morgan gives entirely satisfactory rigorous proofs of their results, in a modern sense. Even De Morgan's combinatorial descriptions were presented as an "analogy" or "explanation," rather than a proof. But all these works were rigorous enough by the standards of their own times: the authors were more interested in the practical utility of their methods than in worrying about situations where the theory might break down.

"Faà di Bruno's formula" was first stated by Arbogast in 1800, and it might as appropriately be named after one of the ten or more authors who obtained versions of it before Faà di Bruno. Only the determinantal formulation of it ought to be called "Faà di Bruno's formula."

Note added in proof. The author has recently become aware of, but has not seen, the following book on Faà di Bruno edited by Livia Giacardi: *Francesco Faà di Bruno. Ricerca scientifica insegnamento e divulgazione* (Studi e fonti per la storia della Università di Torino, vol. 12), Palazzo Carignano, Turin, 2004. On Thomas Knight, more information is to be found in the article by A. D. D. Craik and G. Edwards "In Search of Thomas Knight," *Bulletin Brit. Soc. Hist. Math.* **2** (2004), 17–27.

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